Turbulent Diffusion — A Rigorous Treatment

Pitter Gräff and Dieter Pfirsch

Max-Planck-Institut für Plasmaphysik EURATOM Association, Garching

To Professor Arnulf Schlüter on his 60th Birthday

The problem of passive diffusion by strong turbulence is described by equations of motions of the form $dx/dt = V(t, x)$, $V(t, x)$ being a stochastic velocity field with strong nonlinear dependence on x. Present-day theories are based on certain plausible approximations rather than on a systematic treatment, e.g. in form of a perturbation theory with respect to some smallnes parameters. In this paper a method is described to construct stochastic velocity fields with reasonable velocity distributions and correlations, and to obtain at the same time the exact solution of the corresponding diffusion problem. Brownien motion is a special case of such models. The models can serve to check existing approximate theories, to simulate real turbulence, and to get altogether better insight into turbulent diffusion.

The Problem

The notion of turbulent diffusion is as old a phenomenon as turbulence itself. When Osborne Reynolds observed the onset of turbulence with an indicator dye, the essential feature was an irregular motion of the coloured test particles. Meanwhile there are several other applications of interest that afford a suitable treatment of turbulent diffusion. We give a short — not exhaustive — list.

1. As already mentioned, there is the motion of a small particle within a turbulent fluid. If we are allowed to assume that the particle will immediately follow the fluid, we may identify its velocity with the instantaneous motion of the fluid. This leads to an equation of the form

$$dx/dt = V(t, x),$$

where $V(t, x)$ is the velocity field of the fluid.

2. In a somewhat more refined treatment one should take two additional features into account:

— the finite mass of the particle,

— the fact that besides the stochasticity of the fluid motion (which we accept for the moment without any attempt to explain it itself, e.g. on the basis of a stochastic theory of Navier-Stokes equations) there is present the influence of Brownian motion. We thus expect superpositioning of turbulent and Brownian diffusion. For the motion of a test particle we now expect an equation of the Langevin type [2]:

$$dx/dt = v,$$
$$m \, dv/dt = -\beta(u - V(t, x)) + A(t).$$

Here A is a random stirring force due to the agitation of the temperature-driven collisions and β would in general be a tensor which also takes care of the shape of the particle (friction tensor). We should then have to add further equations for the rotation of the particle. If we assume it is a small rigid sphere, we may omit these additional complications. β then becomes a scalar quantity and we arrive at a rather simple expression in the case of very light particles $(m \to 0)$:

$$dx/dt = V(t, x) + (1/\beta) A(t).$$

This equation is essentially of the same type as that mentioned before in the case of purely turbulent diffusion, if we incorporate the random stirring force within the stochastic velocity field. In the following there is thus no need for a distinction between turbulent and Brownian diffusion [2]. An example of practical application would be in problems of environmental pollution by a chimney stack.

3. In plasma physics, charged particles undergo gyration in the presence of a strong magnetic field, thus being bent to the lines of force. If, however, a small electric field is simultaneously present, we find a drift of the particles according to [3]

$$dx/dt = E \times B.$$

These electric fields are usually due to several kinds of instabilities and it is a convenient assumption that these instabilities produce a rather stochastic

electric field. Formally, our equation is then again of the former type.

4. One may suggest that the penetration of waves through a turbulent medium might be treated in a similar manner. In fact, we have shown that this is correct at least for the eiconal equation. This will be shown elsewhere.

So far; we have provided a description of the task: Given a turbulent fluid $V(t, x)$, what are the transition probabilities of the test particles following the equation of motion? — We should add that at least for two and three dimensions the further assumption that the velocity field is incompressible is very often met (examples 1, 2, 3). In fact, if the medium is air or water, this is fulfilled with a high degree of approximation and it still holds in the plasma case. In the present representation we shall not deal with this question but only remark that in the one-dimensional case incompressibility would not allow any interesting results, whereas the next complicated possibility, two-dimensional flow, which is of interest in the case of a plasma, can be dealt with by a certain variation of the methods to be developed below. This will be the subject of a subsequent paper.

The Conventional Approach

Present-day literature on turbulence deals with the problem to some extent. Monin and Yaglom, for example, have devoted a whole, lengthy chapter (Vol. 1) [4] to it and another reference in book form is given by Csanady [2]. The interest may be due to the fact that official standards for the spreading of, for example, pollutants are based upon acceptable results in this field.

In the older literature [5], the problem was tackled with so-called Lagrangian correlations. It is easily seen that the equation of motion can be integrated along its own path (which, unfortunately, is unknown). Hence, if one introduces the corresponding Lagrangian correlations, some results in terms of them can be achieved. The lack of knowledge of the Lagrangian correlations led to the assumption that they may be identified with some accuracy by the corresponding Eulerian velocity correlations.

Another approach is based upon the assumption that if the velocity field is governed by a Gaussian stochasticity (for its meaning see, for example, Cramer and Leadbetter [6]), then this should also map onto the notion we have of the position of the particle at least for small times as long as a Taylor series may be assumed to be valid. — On the other hand, for very long times it is believed that the correlations of the velocity field may have died out after a finite τ, and hence the path along the velocity field could be considered as a sum of $n = t/\tau$ more or less independent Gaussian parts. For large values of t the central limit theorem would then again predict for x a Gaussian distribution. From these suggestions (where evidently a Lagrangian and a Eulerian picture have been exchanged) one may conclude that the transition probability should be modelled rather acceptably by a Gaussian-like function, which is usually the solution of a suitable, conventional diffusion-type equation with a suitably chosen "turbulent diffusivity". — Some problems are concerned with the superpositioning of Brownian motion.

So far, this cannot be considered as a systematic treatment. The main difficulty stems from the fact that the problem is — at the moment — far from any perturbational treatment. It corresponds to what is called a problem of strong turbulence (for the definition of weak turbulence in contrast compare Pfirsch [7] and Elsässer and Gräff [8] for a treatment). Ten years ago, one of the present authors tried to attack the problem by means of a Dyson-like equation for the transition probabilities. It is possible to obtain a rigorous expression, which unfortunately contains an unknown mass operator as the kernel of the corresponding equation (Gräff) [9]. Despite the fact that it was possible to obtain — by conventional quantum-mechanical perturbation theory — a rigorous equation for the mass operator as well, the result proved to contradict ordinary Brownian motion. The most probable reason for this (unpublished) disaster was that the underlying series, which were rearranged by the perturbation procedure, were not absolutely convergent. This is in keeping with the fact that no plausible smallness parameter can be associated with the problem.

We are thus obliged to seek a rigorous treatment of the problem [10].

Models

A few words on what is meant by "rigorous treatment" are needed. The essential idea is that many

examples of velocity fields which can be dealt with exactly can be given. Furthermore, we shall offer a method of finding additional examples. This may raise the question: Are these examples representative or are they academic?

There is a very famous answer to this question in a similar case. This is the conventional treatment of Brownian motion. Concepts like the Wiener process are introduced. If $x(t)$ is given by a Wiener process, then no finite velocity exists — hence one might hesitate to adopt such a process as a useful tool for describing diffusion from the point of view of Newtonian mechanics. However, the conventional diffusive scaling

$$\langle x^2 \rangle \sim t$$

is incorporated excellently in the Wiener process.

The essential points of the physical reasoning are:

— The process can be subjected to rigorous, mathematical treatment. It thus serves as a ,,reference basis" — even if it is not completely realistic.

— The possibility of deriving a Fokker-Planck equation in many physical situations from a rather fundamental basis proves the Wiener model to be at least in good agreement with what we expect from physics.

The Wiener process is thus considered as kernel of any treatment of classical Brownian motion, thus serving as a reference, as established by, for example, the improvement given by Ornstein and Uhlenbeck.

There is another reason why rigorous results are of interest, even if they look somewhat academic at first glance: They offer a possibility of checking already existing (or subsequently invented) "approximations" under model conditions. For example, in turbulent diffusion the assumption of a finite correlation time of the velocity field is rather crucial to most present-day formulas in order for them to be convincing. However, the case of what we call "frozen-in turbulence" would be a rather outstanding limiting case in which we do not expect any of the existing theories to work. It may thus always be of interest to check whether new suggestions for approximation treatment of turbulent diffusion really implies progress. In particular, they should be compared with their predictions for the frozen-in case. This criterion was proposed to one

of the authors by P. G. Kraichnan on the occasion of a personal discussion [11]. It is precisely the frozen-in case which we are dealing with to some extent in the following.

Our Method

We shall give details explicitly below. Nevertheless, it seems worthwhile to represent the more general argument beforehand in comparison with what has been done elsewhere.

More recently, Lundgren [12] gave a treatment of turbulent diffusion which is based essentially on earlier work by Kraichnan [13]. He deals with the representation of the path of the particles within the fluid in the sense of Lagrange and represents the transition probabilities on nearly the same basis as we shall do. However, he did not consider them as the starting point because in his description the initial conditions (e.g. starting point of the particle) cannot readily be eliminated — which is essential in order to derive the probabilities of the velocity field from the motions of the test particles — irrespective of what special test particle has been chosen. Hence his assumption is the classical one, as formulated by us at the beginning: Given the statistics of the velocity field —, what can be said about the transition probabilities of the test particles? Lundgren subsequently treated a rigorous formula on the basis of certain assumptions (e.g. also the Corrsin hypothesis, which is also the subject of a recent paper by Newman and Herring [14]. We shall derive an equivalent, exact formula, but within a somewhat different, slightly more general frame of consideration which allows a different line of attack.

Given the statistics of all the test particles, what about the stochastics of the velocity field?

In this sense, we are not concerned with the most natural question in physics, the action as a result of cause. Let us assume instead a certain action and ask what was the underlying cause. In practice this means — contrary to Newton — that we do not start with an assumed knowledge about the moving forces and then find the path of a test particle, but start instead with the motion of the particle and subsequently construct "fitting velocity fields".

The essential problem within such a concept is to get rid of the initial starting positions of the par-

ticle. In fact, the instantaneous position of a particle within a moving fluid will be a rather complex function of its initial position:

$$x = f(t, x_0)$$

(Lundgren) [12]. According to this, the distribution of the particles, which is described by their initial statistics together with the transition probabilities of the velocity field, does not look easy to separate into these two parts. Nevertheless, this can easily be done, if one introduces a representation of the paths of the test particles in such a way that the initial conditions are represented very explicitly. This is precisely what we have done.

Implicit Solutions

As already mentioned, we are allowed to start from the equation of motion in the form

$$dx/dt = V(t, x),$$

where $V(t, x)$ is regarded as a stochastic velocity field. This covers the case of Brownian motion, which might be stated explicitly by an equation for the density of test particles of the type

$$\partial c/\partial t + V(t, x) \partial c/\partial x = \tfrac{1}{2} D \Delta c$$

since the Brownian motion can be incorporated into the stochasticity of the velocity field. Hence the latter equation is a consequence of the former and no split between microscopic and macroscopic turbulence will be made.

We write the solution of our equation of motion in the form

$$S(t, x) = \beta$$

where β are constants of motion, e.g. the initial values or certain functions of them. (The arbitrariness involved in this possibility will be discussed below.) Our implicit solution obeys the following equation of motion:

$$(\partial/\partial t + V(t, x) \partial/\partial x) S = 0.$$

Let us consider the number of test particles within a small volume of the β's:

$$dN = g(\beta) d^3\beta.$$

For the corresponding number of test particles within an x-space volume we find

$$dN = g(S(t, x)) \left| \det \partial S_i/\partial x_k \right| d^3x.$$

From this we get an expression for the transition of test particles from an original position x_0 to x within the time t for any special (individual) realisation of the turbulent fluid field, if we start with only one particle at x_0. This corresponds to the following assumption about the initial particle density:

$$n(0, x) = \delta(x - x_0),$$

where $n(t, x) = dN/d^3x$. It immediately follows that

$$g(S(t, x)) = \delta(S(t, x) - S(0, x_0)).$$

This formula applies for any realisation of the field. Statistics is now introduced in a twofold way:

— through the initial conditions, e.g. if $n(x)$ is smeared out: $\langle n(0, x_0)\rangle = n_0(x_0)$,

— through the statistics of the turbulent motion

It seems natural that both sources of statistics are independent.

— A counterexample would be the following:

A stack is assumed to emit pollutants in proportion to the speed of the wind. Then

$$n_0(x) \sim V^2(0, x_0) \delta(x - x_0)$$

and this feedback would have to be taken care of. This is omitted in the following.

Since the above procedure affords the possibility of getting rid of the initial values (as is the aim of any natural science), let us call the above procedure the IF (implicit function) method.

Transition Probabilities

The last remark greatly simplifies the treatment. It is generally adopted and if we denote expectation values by brackets, we get

$$\langle n(t, x)\rangle = \int d^3x_0 \, n_0(x_0)$$
$$\cdot \langle \left| \det \partial S_i/\partial x_k \right| \delta(S(t, x) - S(0, x_0))\rangle$$

The formula describes the distribution of the test particle cloud by means of the corresponding transition probabilities, an expression which can also more explicitly be written in terms of probability distribution for the S-field:

$$\langle \det \left| \partial S_i/\partial x_k \right| \delta(S(t, x) - S(t_0, x_0))\rangle$$
$$= \left\langle \det \frac{\partial}{\partial x_k^{(i)}} \prod_{j=1}^{3} \theta[S_j(t, x^{(j)}) - S_j(t_0, x_0)] \right\rangle_{x^{(j)} = x}$$

Here θ is the Heavyside function. Further transformation yields

$$= \det \frac{\partial}{\partial x_k^{(j)}} \iiint dS_1 \, dS_2 \, dS_3 \iiint dS_1{}^0 \, dS_2{}^0 \, dS_3{}^0 \prod_{j=1}^{3} \theta (S_j - S_j{}^0) \, p_2(t, \, x_i{}^{(j)} \, S_j; t_0, \, x_0 : S_j{}^0; \, j = 1, 2, 3)|_{x^{(j)} = x}$$

or finally

$$= \det \frac{\partial}{\partial x_k^{(j)}} \iiint dS_1{}^0 \, dS_2{}^0 \, dS_3{}^0 \, \mathrm{Prob} \, \{S_j(t, x^{(j)}) > S_j{}^0 | \, t_0, \, y_0 \, p_1(t_0, \, x_0 : S_j{}^0)|_{j=1,2,3; \, x^{(j)} = x}$$

in terms of the conditional probabilities and probability densities.

The Velocity Field

So far, we have derived the statistics of the transition probabilities from those of the S-field. The crucial point is that, parallel to this, we have to get the statistics of the corresponding velocity field on an equal footing. In fact, for the 1-point joint distribution probability we have

$$g(t, x : v) = \langle \delta(v - V(\partial S/\partial t, \nabla S)) \rangle .$$

Here V has to be expressed in terms of S instead of the original t, x. This easily can be done on the basis of the conservation of S along a path

$$\partial S/\partial t + V(\partial S/\partial t; \nabla S) \cdot \nabla S \equiv 0 .$$

The solution of this equation is

$$V = - \det^{-1} | \nabla S_1, \nabla S_2, \nabla S_3 |$$
$$\cdot \sum_{j=1}^{3} \nabla S_l \times \nabla S_m \, \partial S_i / \partial t$$

with i, l, m cyclic.

However, we do not even need this explicit representation in order to calculate $g(t, x : v)$ since we find

$$\delta(v - V)$$
$$= \langle \det | \partial S_i / \partial x_k | \, \delta(\partial S / \partial t + v \cdot \nabla S) \rangle .$$

Again, this can be represented in terms of the probabilities of S:

$$= \left\langle \det \frac{\partial S_i}{\partial x_k} \frac{\partial}{\partial S_i} \prod_{j=1}^{3} \theta \left(\frac{\partial S_j}{\partial t} + v \, \nabla S_j \right) \right\rangle$$
$$= \det \frac{\partial}{\partial v_k^{(i)}} \left\langle \prod_{i=1}^{3} \theta \left(\frac{\partial S_j}{\partial t} + v^{(j)} \, \nabla S_j \right) \right\rangle_{v^{(j)} = v}$$
$$= \det \frac{\partial}{\partial v_k^{(i)}} \mathrm{Prob}$$
$$\cdot \left\{ \frac{\partial S_j}{\partial t} + v^{(j)} \, \nabla S_j > 0, \, j = 1, 2, 3 \right\} \Big|_{v^{(j)} = v} .$$

The latter probability can be completely obtained from the 2-point joint probability of the derivatives of S, which requires that the 5-point distribution in the neighbourhood of t, x be known (for any component of S):

$$\int_0^{\infty} d\lambda \iiint \delta(S_t + v \, \nabla S - \lambda)$$
$$\cdot p(t, x : S_{x_1}, S_{x_2}, S_{x_3}) \, dS_t \, dS_{x_1} \, dS_{x_2} \, dS_{x_3} .$$

The main result thus is: Given the statistics of S, we can derive the statistics of the velocity field as well as the corresponding transition probabilities simultaneously and in parallel. It may therefore be convenient to denote S as some kind of a "stochastic potential". One may observe that we have not treated the problem in the conventional way, where one would primarily have in mind given statistics for the v-field.

We wish to add the trivial remark that the two-point velocity distribution can, of course, be obtained in a similar way by treating the expression

$$\langle \delta(v_1 - V(t_1, x_1)) \, \delta(v_2 - V(t_2, x_2)) \rangle .$$

Restriction on S. — The Gauge Group

We have to establish a unique connection between $S(t, x)$ and its initial values β. This can only be done if

$$\det \partial S_i / \partial x_k \neq 0, \, \pm \infty .$$

This restriction introduces several complications from the practical point of view since it rules out, for example, Gaussian stochasticity for the S-field itself. This means that one of the best understood stochastic fields is not available for our treatment.

Furthermore, since S was denoted as a stochastic potential, we expect it to contain some redundant information. This is in fact correct, as can be seen by the fact that the essential information within our chosen set of constants is not destroyed if we apply a $1-1$ unique transformation $\beta \to \hat{\beta}$, or $S \to \hat{S}$.

Again we would have to require that

$$\det \partial \hat{S}_i / \partial x_k \neq 0, \quad \pm \infty.$$

We expect the same results within the $\hat{S}$-representation as we had already for the S-representation. In fact, in order to check this explicitly, we have to show that the velocity field as well as the transition probabilities are left unchanged. The velocity field is obtained from

$$\frac{\partial \hat{S}_i}{\partial t} + \boldsymbol{V} \frac{\partial}{\partial \boldsymbol{x}} \hat{S}_i = \sum \frac{\partial \hat{S}_i}{\partial S_l} \left(\frac{\partial S_l}{\partial t} + \boldsymbol{V} \frac{\partial S_l}{\partial \boldsymbol{x}} \right).$$

Hence the left-hand side vanishes together with

$$\frac{\partial S_l}{\partial t} + \boldsymbol{V} \frac{\partial S_l}{\partial \boldsymbol{x}} = 0$$

if $\det \partial \hat{S}_i / \partial S_l$ is finite. Conversely, the same holds if the inverse expression is finite:

$$\det \partial \hat{S}_i / \partial S_l \neq 0.$$

Under this condition the densities $n(t, \boldsymbol{x})$ also remain invariant: From

$$\frac{\partial \hat{S}_l}{\partial x_k} = \sum_l \frac{\partial \hat{S}_i}{\partial S_l} \cdot \frac{\partial S_i}{\partial x_k}$$

we find

$$\det \left(\frac{\partial \hat{S}_i}{\partial x_k} \right) = \det \left(\frac{\partial \hat{S}_i}{\partial S_l} \right) \cdot \det \left(\frac{\partial S_i}{\partial x_k} \right).$$

Using this within the identity

$$\delta(\boldsymbol{S} - \boldsymbol{S}(x_0, 0)) \, d^3 S$$
$$= \delta(\hat{\boldsymbol{S}} - \hat{\boldsymbol{S}}(0, x_0)) \, d^3 \hat{S}$$
$$= \delta(\hat{\boldsymbol{S}} - \hat{\boldsymbol{S}}_0) \det(\partial \hat{S}_i / \partial x_k) \, d^3 S$$

we get

$$\delta(\hat{\boldsymbol{S}} - \hat{\boldsymbol{S}}_0) = \frac{1}{\det(\partial S_i / \partial S_k)} \delta(\boldsymbol{S} - \boldsymbol{S}_0).$$

Therefore

$$\left\langle \det \frac{\partial \hat{S}_i}{\partial x_k} \delta(\hat{S} - \hat{\boldsymbol{S}}_0) \right\rangle = \left\langle \frac{\det \dfrac{\partial \hat{S}_i}{\partial x_k} \delta(\boldsymbol{S} - \boldsymbol{S}_0)}{\det \dfrac{\partial \hat{S}_i}{\partial S_l} \delta(\boldsymbol{S} - \boldsymbol{S}_0)} \right\rangle$$
$$= \left\langle \det \frac{\partial S_i}{\partial x_k} \delta(\boldsymbol{S} - \boldsymbol{S}_0) \right\rangle$$

and n remains invariant.

The essential result is: There is a whole group of transformations $\boldsymbol{S} \rightarrow \hat{\boldsymbol{S}}$ under which the physical properties are conserved. This means that there exist different stochastic representations for $\boldsymbol{S}$ which end up with the same result for the stochasticity of the velocity field and its diffusion probabilities. This corresponds to what we want to call a stochastic gauge group.

Causal Transformations

For completeness we want to mention the following point: If we apply, for example, a (causal) transformation onto $\boldsymbol{x}$ or t, the corresponding change within the different formulas can be made readily obvious. According to this we get together with any solved problem a whole family of additional rigorously treated problems. In comparison with already existing approximation formulas, this will be useful because the approximation will not be expected to show the same "invariance" as the rigorous result.

The 1-Dimensional Case

In the following we shall treat several examples which refer to this case. In fact, since our method in its present form is not adapted to the problem of incompressibility, in case of higher dimensions we would always imply a certain amount of stochastic compressibility of the fluid, whereas in one dimension the assumption of incompressibility would only allow trivial solutions.

Merely for reference, we give the simplified results for our case somewhat explicitly.

For the velocity field we find

$$g(t, x : v) = \frac{\partial}{\partial v} \text{Prob} \left\{ \frac{\partial S}{\partial x} + v \frac{\partial S}{\partial t} > 0 \right\}$$

according to

$$v = - S_t / S_x$$

and for the transition probabilities

$$\boldsymbol{p} \{t : x \,|\, t_0 : x_0\}$$
$$= \langle |\, \partial S / \partial x \,| \, \delta(S(t, x) - S(t_0, x_0)) \rangle$$
$$= \frac{\partial}{\partial x} \int dS_0 \, \text{Prob} \{S(t, x) > S_0 \,|\, t_0, x_0 : S_0\}$$
$$\cdot p_1(t_0 x_0 : S_0).$$

The gauge group means that for any fixed t the $S(x)$ is a monotonic function (either decreasing or increasing), and that any (fixed) causal, monotonic mapping is allowed (which evidently does not break the monotonicity).

Jumps, Trapping and Boundary Conditions

So far, we have assumed that $S(t, x)$ defines a unique mapping between the whole set of initial values and their present ones. Correspondingly, we had S as a strictly monotonic function on the total x-axis. However, these conditions may be weakened:

— If S_x becomes infinite in certain points, these correspond to the endpoints of certain intervals which behave similarly to the whole axis as before: We now expect a map of this interval into itself.

— Within any such interval (for any fixed t) we expect S to increase or decrease monotonically, but we may omit the strict monotonicity. In order to avoid a break in 1–1 uniqueness, we are then obliged to assume that for any part where S does not increase it is not defined only the right endpoint, for example, being associated with S. Such motions will correspond to jumps of the test particles. In the case of frozen-in turbulence, we shall find a simple example. It shows that our treatment thus also deals with the possibility of, for example, Poisson processes within the same kind of framework.

Finally, it may be stated that — as is common in Brownian motion — we may admit the possibility of absorption (e.g. chemical reaction), reflexion and transmission of test particles at certain points or certain boundaries. In principle, these boundaries may not be identified with those of the fluid itself. The present paper will not be concerned with these latter questions.

One-Dimensional Examples

1. Brownian Motion

We give the derivation merely for completeness in order to show that the new treatment incorporates the older results. Let

$$S = x - \sigma(t),$$

where σ is defined essentially by the Wiener process:

$$\dot{\sigma} = \sqrt{D}\,\xi(t), \quad \sigma(0) = 0.$$

Here ξ is a Gaussian, stochastic function with the variance

$$\langle \xi(t)\,\xi(t') \rangle = \delta(t - t')$$

and vanishing mean value (to be more precise, it is a stochastic distribution; see Gelfand and Schilow [15]). The probability density obeys

$$\frac{\partial p}{\partial t} = \frac{D}{2} \frac{\partial^2 p}{\partial \sigma^2}, \quad p(0, \sigma) = \delta(\sigma)$$

with the solution

$$p(t : \sigma) = \frac{1}{\sqrt{2\pi D t}} \exp\{-\sigma^2/2Dt\}, \quad t > 0.$$

Starting from a given initial value, we find

$$x - \sigma(t) = x_0$$

and hence immediately for the transition probability

$$\boldsymbol{p}(t : x \mid x_0) = \frac{1}{\sqrt{2\pi D t}} \exp\{-(x - x_0)^2/2Dt\}$$

a result which may also be obtained more formally from our formulas:

$$\text{Prob}\,\{S(t, x) > S_0 \mid t_0, x_0 : S_0\}$$
$$= \text{Prob}\,\{t, x : \sigma < x - x_0\} = \int_{-\infty}^{x - x_0} p(t : \sigma)\,\mathrm{d}\sigma$$

with

$$p(t_0, x_0 : S_0) = \delta(S_0 - x_0).$$

Hence

$$\boldsymbol{p}(t; x \mid x_0) = \langle\,|\,\partial S/\partial x\,|\,\delta(S(t, x) - S(t_0, x_0))\rangle$$
$$= (\partial/\partial x) \int_{-\infty}^{x - x_0} p(t : \sigma)\,\mathrm{d}\sigma$$
$$= p(t : x - x_0).$$

The density of the test particles after a time t will therefore be

$$\langle n(t, x) \rangle = \int_{-\infty}^{+\infty} n_0(x_0)\,p(t : x - x_0)\,\mathrm{d}x_0$$

which is again a solution of the diffusion equation. The formulas show very explicitly that in such a treatment all test particles feel the same stochastic driving force, irrespective of their starting position, and so they all run more or less "parallel" within any individual realisation of the fluid motion. This is a somewhat unrealistic point in the conventional treatment of Brownian motion. It could only be avoided if we also admit stochastics on x of the driving fields. This is now possible.

2. A Case of "Frozen-in" Turbulence

Frozen-in turbulence means that the correlation time for the fluid motion is rather long (idealized

limit $\tau \to \infty$). It is natural to assume besides that no Brownian motion should be present. We thus expect

$$S = \varrho(x) - t,$$

where $\varrho(x)$ is a monotonically growing or decreasing function. Examples of this kind are of primary interest because they are far from any — even approximate — treatment till now. As a first example let us assume

$$\varrho(x) = \varepsilon(x) + \int_0^x \varphi(x')\, w^{2n}(x')\, dx', \quad \varepsilon(0) = 0,$$

where $\varepsilon(x)$ is a monotonically growing function, φ is positive, n is an integer and w is the Wiener process:

$$dw/dx = \xi(x), \quad \langle \xi \rangle = 0,$$

$$\langle \xi(x)\,\xi(x') \rangle = \delta(x - x').$$

By very conventional methods (e.g. P. Levy [16]) one can easily show that the probability density for ϱ and w fulfills the following equation:

$$\frac{\partial p}{\partial x} + (\varepsilon'(x) + \varphi(x)\, w^{2n}) \frac{\partial p}{\partial \varrho} = \frac{1}{2} \frac{\partial^2 p}{\partial w^2},$$

and we are interested in solutions according to the initial condition

$$p(x = 0;\, \varrho, w) = \delta(\varrho)\, p_0(w).$$

In fact, if we admit for $p_0(w)$ not merely the Wiener assumption $\delta(w)$, this is merely a trivial generalisation of the original Wiener version, which may be useful later on.

The transition probabilities cannot be obtained without solving this equation. This will not be done in this paper despite the fact that for $n = 1$ an analytical treatment is possible. Instead, we shall try to get some of the interesting information, e.g. the essentials of the spread out of test particles, by a somewhat simpler method of averaging. However, mainly for purposes of reference, we shall give a formula for the transition probabilities, starting from

$$\mathrm{Prob}\,\{S(t, x) > S_0 \mid t_0 = 0, x_0 : S_0\}$$
$$= \mathrm{Prob}\,\{\varrho(x) > t + S_0 \mid x_0 : S_0\}$$
$$= \int_{t+S_0}^{\infty} d\varrho \int_{-\infty}^{+\infty} dw\, p(x : \varrho, w).$$

This yields

$$\partial/\partial x\, \mathrm{Prob}\,\{S(t, x) > S_0 \mid x_0 : S_0\}$$
$$= \int_{-\infty}^{+\infty} dw(\varepsilon' + \varphi\, w^{2n})\, p(x : t + S_0, w)$$

and we finally get

$$\langle n(t, x) \rangle = \int_{-\infty}^{+\infty} dx_0\, n_0(x_0) \int_{-\infty}^{+\infty} dS_0\, p(t_0, x_0 : S_0)$$
$$\cdot \int_{-\infty}^{+\infty} dw(\varepsilon' + \varphi(x)\, w^{2n})$$
$$\cdot p(x;\, t + S_0, w).$$

Later on we shall offer a particular generalization and this result then will hereafter be cited as "old".

Scaling Laws

Let us consider

$$\langle f(S(t, x)) \rangle = \langle f(S(t_0, x_0)) \rangle.$$

Herein the averages may be taken for fixed values of

$$t = \bar{t} \quad \text{and} \quad x = \bar{x}.$$

The above mean values can then be interpreted as a relation for allowed and corresponding values of $\bar{t}$ and $\bar{x}$. For a suitable choice of f this may prove to be a tool for obtaining information about the spread out of our test particles. In principle, the function f may be allowed to depend upon the individual realization — hence it might be a stochastic process itself.

In our case of frozen-in turbulence, we may choose for f the identity. For any particle, starting initially at the origin, we find

$$\langle S(\bar{t}, \bar{x}) \rangle = 0$$

or

$$\bar{t} = \langle \varrho(\bar{x}) \rangle$$
$$= \varepsilon(\bar{x}) + \int_0^{\bar{x}} \varphi(x) \langle w^{2n}(x) \rangle\, dx$$
$$= \varepsilon(\bar{x}) + 1.3 \ldots (2n - 1)$$
$$\cdot \int_0^{\bar{x}} \varphi(x)\, x^n\, dx, \quad \text{if} \quad W = 0.$$

An expectation of this type corresponds very clearly to an average time $\langle t \rangle$, according to the dispersion of the „times of arrival" of different test particles within different realizations at a fixed point $\bar{x}$. Averages of this kind are also familiar from the theory of Brownian motion, where they lead to a

connection with potential theory. For the connection with the dwell-time see the appendix.

In a simple case such as

$$\varepsilon = \varepsilon_0 = \text{const},$$

$$\varphi = \varphi_0 = \text{const}$$

we obtain

$$\bar{t} = \varepsilon_0 + \frac{(2n-1)\ldots 3.1}{n+1}\,\varphi_0\,\bar{x}^{n+1}\,.$$

Hence we recover the conventional diffusion law for long times in the case $n = 1$:

$$\lim_{\bar{t}\to\infty} (\bar{x}^2/\bar{t}) = \text{const}\,.$$

It would not be valid, however, in any other case $n > 1$, which illustrates that for frozen-in turbulence the classical scaling law of diffusion may break down.

The Velocity Distribution of Example 2

Fortunately, in this connection we do not need to know the whole probability distribution for ϱ and w, as was necessary for the transition probabilities. In fact, instead of using our general formulas, we may return directly to the definition of v in terms of S. We then immediately find

$$g(t, x : v) = \int \delta\left(v - \frac{1}{\varepsilon'(x) + \varphi(x)\,w^{2n}}\right)$$
$$\cdot \hat{p}(x : w)\,dw\,.$$

This shows that all that is required is the distribution of w alone, which is nearly the 1-point distribution of the Wiener process. In fact, one has

$$\hat{p}(x : w) = \int p(x : \varrho, w)\,d\varrho$$

with

$$p(x > 0, \varrho = 0, w) \equiv 0\,.$$

Hence we do not have to solve completely for p but find

$$\frac{\partial \hat{p}}{\partial x} = \frac{1}{2}\,\frac{\partial^2 \hat{p}}{\partial w^2}$$

with

$$\hat{p}(x = 0, w) = p_0(w)\,.$$

The solution is just

$$\hat{p}(x : w) = \int_{-\infty}^{+\infty} \frac{\exp\{-(w - w_0)^2/2\,x\}}{\sqrt{2\pi x}}\,p_0(w_0)\,dw\,.$$

This may be inserted into the expression for g, yielding

$$g(t, x : v) = \left\{p\left(x : \left[\frac{1}{\varphi}\left(\frac{1}{v} - \varepsilon'\right)\right]^{1/2n}\right)\right.$$
$$+ p\left(x : -\left[\frac{1}{\varphi}\left(\frac{1}{v} - \varepsilon'\right)\right]^{1/2n}\right)\right\}$$
$$\cdot \frac{d}{dv}\left(\frac{1}{\varphi}\left(\frac{1}{v} - \varepsilon'\right)\right)^{1/2n}$$

for

$$0 < v < \frac{1}{\varepsilon'(x)}\,.$$

In the case $n = 1$, when we expect classical diffusion, we find especially for

$$\varepsilon = 1\,, \quad \varphi = 1\,,$$

$$p_0(w_0) = \delta(w_0 - W)$$

a velocity distribution which shows a slightly divergent behaviour at the end of the interval. Graphs are given in the figures for different values of x and W.

3. A Case of Homogeneous Frozen-in Turbulence

Let us consider

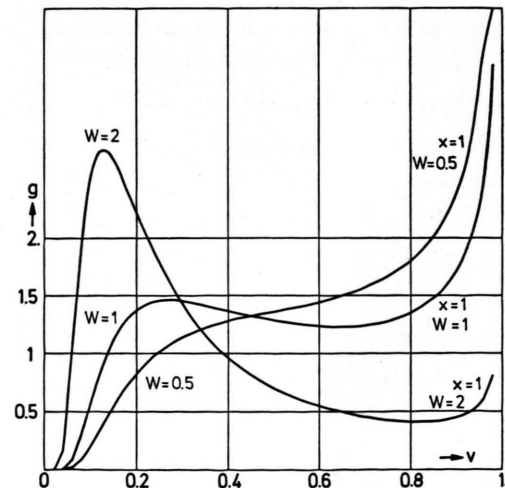$$\varrho(x) = \int_0^x \frac{dx'}{\varepsilon + u^2(x')}$$
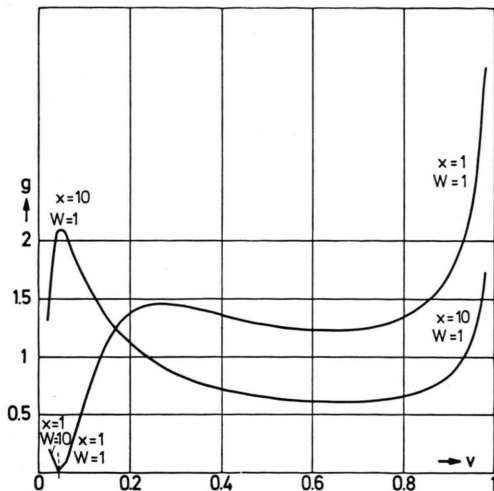

Fig. 1. Velocity distributions of example 2 for fixed x.

Fig. 2. Velocity distributions of example 2 for fixed W.

on the assumption that $u(x)$ is a homogeneous stochastic process, e.g. a Gaussian one, which is completely characterized by its mean value

$$\langle u(x)\rangle = 0$$

and covariance (correlation function)

$$\langle u(x)\, u(y)\rangle = C(x-y).$$

For the velocity field we obtain

$$v = \varepsilon + u^2(x).$$

Hence the corresponding mean velocity will be

$$\langle v\rangle = \varepsilon + C(0)$$

irrespective of the position. The correlation function of v can also be easily calculated by means of the cluster property for Gaussian distributions:

$$\langle v(x)\, v(y)\rangle - \langle v\rangle^2 = 2\, C^2(x-y).$$

In order to obtain the scaling law for the dispersion of test particles, we use the ASUCON-method (see the appendix).

$$\langle t\rangle = \int_0^{\bar{x}} \mathrm{d}x' \langle 1/(\varepsilon + u^2)\rangle.$$

The integrand is a constant, hence we find the result

$$\bar{x}/\bar{t} = \langle 1/(\varepsilon + u^2)\rangle^{-1}.$$

Here the expectation value of interest,

$$\left\langle \frac{1}{\varepsilon + u^2}\right\rangle = 2\int_0^\infty \frac{\exp\{-u^2/2\,C(0)\}}{\sqrt{2\,\pi\,C(0)}}\,\frac{\mathrm{d}u}{\varepsilon + u^2}.$$

can be rigorously calculated in terms of the error function (Gradstein et al. [17]). For our purpose an approximate estimate as is given in Abramowitz et al. [18] is completely sufficient. We obtain

$$\frac{\bar{x}}{\bar{t}} = \frac{\varepsilon + \sqrt{\varepsilon^2 + 2\,a\,C(0)\,\varepsilon}}{2},$$

where a is some constant between $4/\pi$ and 2.

The essential feature is: We do not recover the conventional diffusion law. In the case $\varepsilon \to \infty$ this is what one might have expected since the stochastic part does not play any essential role. In fact, the corresponding limit can easily be found to be

$$\bar{x}/\bar{t} \approx \langle v\rangle$$

in accordance with the trivial physical situation. Of greater interest, however, is the case $\varepsilon \to 0$, which is dominated by the stochastic properties. Here we find

$$\bar{x}/\bar{t} \sim \sqrt{\varepsilon\langle v\rangle}$$

which demonstrates the strong deviation of this case of frozen-in turbulence from the conventional results. This may be of particular interest in comparing the Lagrangian and Eulerian treatments, when often the one method (Lagrange) would yield the correct result if its correlations were allowed to be identified with the Eulerian ones.

Merely for completeness, we add the whole 1-point distribution function for the velocities, which can easily be found on the basis that it is Gaussian for u:

$$g(t, x : v) = \begin{cases} \dfrac{\exp\{-(v-\varepsilon)/2\,C(0)\}}{\sqrt{2\,\pi\,C(0)}}\cdot\dfrac{2}{\sqrt{v-\varepsilon}} \\ \hfill v \geqq \varepsilon, \\ 0 \hfill \text{otherwise}. \end{cases}$$

It is also possible to give a rigorous formula for the transition probability if we assume that the correlation function decays exponentially [19]:

$$C(x-y) = c\exp\{-\beta\,|x-y|\}.$$

The process $u(x)$ could then be considered as "white noise" passing through a filter, and we obtain the following Fokker-Planck equation for $p(x : \varrho; u)$:

$$\frac{\partial p}{\partial x} + \frac{1}{\varepsilon + u^2}\frac{\partial p}{\partial \varrho} - \beta\frac{\partial u\,p}{\partial u} = \frac{c}{2}\frac{\partial^2 p}{\partial u^2}$$

from which the transition probability of interest results essentially after solving and integration over u.

4. Jumps

In order to ensure continuity of our test particle motion, we had forbidden dS/dx to vanish. If, however, we restrict ourselves to „weak monotony", this would perhaps allow an occasional zero for the the derivative within a whole interval of x. Correspondingly, this would imply a jump of the particle. (If the jump occurs in velocity space, this would be described as a collision.)

The situation can be ilustrated if we again consider a case of frozen-in turbulence. Let

$$M(x) = \max_{0 \leq x' < x} w(x')$$

be the "maximum Wiener Process" (MWP). If we identify $\varrho(x)$ with $M(x)$, the transition probabilities can be obtained from

$$p(x:M) = \begin{cases} 2\,p_\omega(x:M) & M \geqq 0\,, \\ 0 & M < 0\,, \end{cases}$$

where w refers to the conventional Wiener process. Conversely, we have (e. g. using formula (*) of the appendix):

$$p(\varrho:x) = (\varrho/\sqrt{2\,\pi})\exp\{-\varrho^2/2\,x\}\,1/x^{3/2}$$

and it can be further shown that the process $x(t)$ has essentially the character of an increasing Poisson process, e.g. a sum of discrete jumps.

Any sample path thus looks in time like a steeplechase in which the horses have to jump over a sequence of arbitrarily arranged but locally fixed hurdles of different widths. Hence we may speak of the steeplechase process.

If instead of the Wiener process we had used within the above procedure any other continuous process without monotonic growth, we would also have ended up with other types of steeplechases.

Generalized Brownian "Movements"

In the conventional treatment of Brownian motion two test particles starting at different locations are considered either as being completely independent or as travelling parallel if they are subjected to the same stochastic force

$$dx \sim dw(t)\,,$$

where $w(t)$ is the Wiener process. However, the proportionality should, in principle, be a random function of the position x with a finite correlation length. Otherwise we would never be able to predict the diffusion of an initial pattern of test particles.

The problem is within our treatment if we allow

$$S(t, x) = \varrho(x) - \sigma(t)\,,$$

where $\varrho(x)$ and $\sigma(t)$ are stochastic functions, which for simplicity we shall assume to be independent:

$$p(t, x : \varrho, \sigma) = p^{(\varrho)}(x : \varrho)\,p^{(\sigma)}(t : \sigma)\,.$$

Referring to our previous cases, we find

$$\langle n(t, x) \rangle = \int_{-\infty}^{+\infty} d\sigma\,p(t : \sigma)\langle n_{old}(\sigma, x) \rangle$$

for the transition probabilities. From

$$v = \dot\sigma(t)/\varrho'(x)$$

we also conclude that

$$g(t, x : v) = \int_{-\infty}^{+\infty} d\dot\sigma\,p(t : \dot\sigma)\,g_{old}(t, x : v/\dot\sigma)\,1/\sigma\,.$$

If we choose for $\hat p(t) = w(t)$ a Wiener process in time, this leads to the above-mentioned, partially coherent Brownian "movement" with spatial correlations. Let us consider an example:

Brownian Jumping

If $\varrho(x)$ is essentially the maximum Wiener process

$$\varrho(x) = \begin{cases} M_1(x) & x \geqq 0\,, \\ -M_2(-x) & x \leqq 0\,, \end{cases}$$

where $M_{1,2}$ may refer to independent (or dependent) Wiener processes $W_{1,2}(x)$, we obtain for the transition probability from the origin

$$\boldsymbol{p}(t, x > 0 \mid t_0 = 0, x_0 = 0)$$

$$= \int_0^\infty \frac{\exp\{-\varrho^2/2\,t\}}{\sqrt{2\,\pi\,t}}\,\frac{\exp\{-\varrho^2/2\,x\}}{\sqrt{2\,\pi\,x}}\,\frac{\varrho}{x}\,d\varrho\,.$$

The integration can easily be done yielding

$$\boldsymbol{p}\,dx = \frac{1}{2\,\pi}\,\frac{1}{\sqrt{x/t}\,(1 + x/t)}\,\frac{dx}{t}\,.$$

This result is in agreement with our scaling law: $\langle \varrho^2(\bar x) \rangle = \langle \sigma^2(\bar t) \rangle$. The difference between ordinary diffusion and this enhanced rate is due to the fact that the steps are now small but finite since the steeplechase process now continually reverses direc-

tion in time. Owing to this property we may call the process „Brownian hurdle jumping" (BHJ) or simply "Brownian jumping".

Using

$$S = \varrho^m(x) - \sigma^n(t)$$

we are obviously also able to construct other scaling laws of diffusion (leading accordingly to BHJ (m, n) processes).

Turbulent Drift and Diffusion

In the case of frozen-in turbulence we are really dealing with a somewhat irregular shift of our test particles in one direction since "the wind always blows from the same side", so to speak. This situation completely changes when extended to "turbulent Brownian motion" with its permanent change of wind direction and strength. It would thus be better to indicate frozen-in turbulent transport as "turbulent drift" and denote turbulent Brownian motion as real diffusion.

The difference can easily be explained if we choose for $\sigma(t)$ an extremely simple model:

$$\sigma = a f(t).$$

Here it is only the parameter a that is a simple stochastic variable, whereas $f(t)$ may be any causal function. We than find

$$\langle n(t, x) \rangle = \int da\, p(a) \langle n_{\text{old}}(x, a f(t)) \rangle,$$

$$g(t, x : v) = \int da\, p(a)\, g_{\text{old}}(t, x : v/a f(t))\, 1/a f.$$

The latter formula is of interest with respect to the possibility of "smearing out" our previous singularities in the velocity distribution (see figures).

Naively trying the ASUCON method, we find

$$\langle S(\bar{t}, \bar{x}) \rangle = \langle \varrho(\bar{x}) \rangle - \langle a \rangle f(\bar{t})$$

which in the case of vanishing mean $\langle a \rangle$ would not yield any useful result for the conditions, and hence no scaling law could be extracted in this way. The reason is evidently that particles which drift to the right within one realization are compensated in the average by others which drift to the left within another realization. We thus expect what is conventionally called diffusion. Of course, in the above case it is obvious that the corresponding scaling law should be the same as for any individual realization, which means that it should be that of turbulent drift.

Is there another possibility of obtaining this result on the basis of our ASUCON method more directly? Several suggestions may be offered.

1. Let us consider

$$\langle a S \rangle = \langle a \varrho(\bar{x}) \rangle - \langle a^2 \rangle f(\bar{t}).$$

Starting with particles from the origin $(S = 0)$, this yields a relation of the required type between $\bar{t}$ and $\bar{x}$.

2. In a similar way, division by a might be helpful if $\langle 1/a \rangle$ is finite.

3. $\langle S^2 \rangle$ can be used for a more complicate estimate. Especially in the frozen-in case starting at the origin, we may use directly

$$\langle \varrho^2(\bar{x}) \rangle = \langle a^2 \rangle f^2(\bar{t}).$$

The latter may be demonstrated in the case of our frozen-in homogeneous turbulence when $f(t) = t$:

$$\langle \varrho^2(\bar{x}) \rangle = \int_0^{\bar{x}} \int \frac{du_1\, du_2\, p(x_2 - x_1 : u_1, u_2)}{(\varepsilon + u_1^2)(\varepsilon + u_2^2)}.$$

For $x \gg 1/\beta$, the correlation length of $u(x)$, the integral essentially factorizes and we obtain

$$\sim \langle 1/(\varepsilon + u^2) \rangle^2 x^2 - \text{terms}\,(\sim x)$$

which yields for the diffusion essentially the same law as we had before for the case of turbulent drift.

If $\langle a \rangle$ is assumed not to vanish, we expect a mixture of drift and diffusion, e.g.

$$\langle v(t, x) \rangle = f(t) \langle a \rangle \langle v(t, x) \rangle_{\text{old}}$$

and similarly for the correlation function

$$\langle v(t, x)\, v(t', x') \rangle = f(t) f(t') \langle a^2 \rangle C_{\text{old}}.$$

This allows an interesting conclusion: The dependence of $\langle n \rangle$ or g on a is much more complicated. Hence for the same values of $\langle a \rangle$ and $\langle a^2 \rangle$ rather different results for the transition probability are to be expected, whereas the velocity mean and correlation may be the same: It is not at all possible to determine turbulent diffusion or drift with just a knowledge of the lowest moments of the velocity field.

Leibniz Transformation

If we want to remove the vanishing probabilities at $v = 0$ in our "old" version of g, we may try to smear

out the distributions by suitable transformation. In cosmology the Leibniz transformation was introduced by shifting the coordinate system with arbitrary velocities [20]. By analogy, we set

$$S = \varrho(x - \lambda(t)) - \sigma(t)$$

and find

$$v = \dot{\lambda} + \frac{\dot{\sigma}}{\varrho'(x - \lambda(t))}.$$

Hence our old g ought to fold with, for example, the probability for λ, which allows the required effect. This is correct even for the case

$$\lambda(t) = \lambda \cdot t$$

of a random Galilean transformation.

Extensions

Already existing rigorous examples may serve as a tool for the construction of more complicated cases, e. g. those by no means separable in x, t. Let φ be a monotonically growing function of two variables

$$\varphi(\xi_1, \eta_1) \leqq \varphi(\xi_2, \eta_2),$$

where

$$\xi_1 \leqq \xi_2, \quad \eta_1 \leqq \eta_2.$$

Let us assume that S_1 and S_2 are "normalized" as monotonically growing. Then

$$S = \varphi(S_1, S_2)$$

may again serve as an interesting candidate for turbulent diffusion if the S_i were so already. In fact, if we are allowed to assume that

$$\text{Prob}\{S_i(t, x) > S_i{}^0 \mid S_i(t = 0, x_0) = S_i{}^0\}$$

is known for $i = 1, 2$ and we further assume that the two are statistically independent, we are able to calculate

$$\text{Prob}\{\varphi > S^0 \mid \varphi(S_1(0, x_0), S_2(0, x_0)) = S^0\}$$

on the basis of a suitable "folding" procedure. Hence the transition probabilities are available. (Again, the ASUCON method may sometimes be helpful.) For the velocity field we obtain a convex superposition:

$$v = \frac{v_1 + \Gamma v_2}{1 + \Gamma},$$

where

$$\Gamma(t, x) = \frac{\varphi_{S_2} S_2'}{\varphi_{S_1} S_1'} \geqq 0.$$

Let us consider some trivial examples. We have treated several cases of frozen-in turbulence. Let us consider two monotonically increasing functions f and g and let

$$S = f(\varrho_1(x) - t) + g(\varrho_2(x) - t)$$

where the ϱ_i's refer to different cases of frozen-in turbulence. The new velocity field becomes time-dependent and varies between the old values $v_1(x)$ and $v_2(x)$ in the course of time.

In a similar way, we may generalize the Leibniz transformation by, for example,

$$S = \varrho_1(x - t) + \varrho_2(2x - t).$$

These are merely simple examples of a wide variety of possibilities for illustrating the fact that our originally of necessity restricted examples may serve as a starting point even for the treatment of rather involved situations and to serve as some kind of a "reference system" for the latter.

Further details will be left to a forthcoming paper.

Appendix

The ASUCON Method

In order to obtain some insight into the meaning of the $\bar{t}, \bar{x}$ introduced in connection with the scaling laws, we may explicitly denote by subscripts any condition which is kept fixed when averaging. Then the average "sub conditione" (ASUCON) reads

$$\langle f(S(\bar{t}, \bar{x})) \rangle = \langle f(S) \rangle_{\bar{t}, \bar{x}, f}.$$

In any conventional treatment of diffusion one solves explicitly for x before averaging and then looks for an expression of the type

$$\langle f_1(x(t)) \rangle = \langle f_1(x) \rangle_{t, S, f_1}$$

as a function of time. — Again, this is a certain ASUCON. The most common form of f_1 is a square. Is there any connection between the above two kinds of average?

Let us assume that any realisation of $S(t, x)$ grows monotonically (this can always be achieved by an equivalence transformation). One then has

$$\text{Prob}\{S(t, x_1) > S_1\} = \text{Prob}\{x(t, S_1) \leqq x_1\},$$

where $x(S, t)$ is the "inverse function" at any instant t. With the corresponding probability densities this yields

$$\int_{S_1}^{\infty} p^{(s)}(t, x_1 : S') \, dS' = \int_{-\infty}^{x_1} p^{(x)}(t, S_1 : x') \, dx'.$$

Applying the second derivative with respect to x and S, we find

$$\partial p^{(s)}/\partial x_1 + \partial p^{(x)}/\partial S_1 = 0.$$

Multiplication by S and integration yields

$$\partial \langle S(t, x)\rangle/\partial x = \int p^{(x)}(t, S' : x) \, dS'.$$

Interpretation becomes especially clear in the case of frozen-in turbulence, when we obtain

$$\frac{\partial p^{(p)}(x : \varrho)}{\partial x} + \frac{\partial p^{(x)}(\varrho : x)}{\partial \varrho} = 0 \qquad (*)$$

from which follows that

$$\partial/\partial x \langle \varrho(x)\rangle = \int p^{(x)}(\varrho' : x) \, d\varrho'.$$

The left-hand side has a simple meaning: it measures the dwell time $d\tau$ of our test particle within the small interval dx:

$$dx \langle \varrho'(x)\rangle = \langle dx/v(x)\rangle.$$

Hence, integrating over a finite interval, we obtain

$$\tau(x) = \langle \varrho(x)\rangle - \langle \varrho(0)\rangle$$
$$= \int \text{Prob}\{x(\varrho') \in [0, x)\} \, d\varrho'.$$

We therefore find our previous $\bar{t}$ to be essentially identical to the dwell time, e.g.

$$\bar{t}(\bar{x}) = \tau(\bar{x})$$

if the particle starts at the origin and ϱ vanishes there.

From this interpretation it is clear that the dwell time (as a measure of the confinement time within a region in x-space) and the diffusion rate (= drift from 0 to x) are the same, at least in one dimension, as are their scaling laws.

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